International Baccalaureate

Higher Level Mathematics Exploration

Properties and applications of the generalized distance between two points

Number of pages: 14

Introduction

Throughout my experience of learning mathematics, I read various additional materials beyond the textbook. Once I was amazed by a concept called generalized distance. It introduces new ideas of distances that have real-world implications, even involving chess and city traveling. Therefore, I'm urged to explore some properties and applications of the generalized distance between two points.

Definition of key terms

The following definitions all exist only in Euclidean geometry. Non-Euclidean geometry is too complicated and definitely beyond the scope of this IA, so I won't discuss that.

D refers to the distance between two points specified by each problem.

Points $A(x_1, y_1)$, $B(x_2, y_2)$ are in a two-dimensional plane.

$\Delta x = x_1 - x_2$ refers to the difference of x-coordinates between A and B.

$\Delta y = y_1 - y_2$ refers to the difference of y-coordinates between A and B.

Define the "Minkowski distance of order m (m-norm distance)" (Distance - Wikipedia, 2021) between A and B as "L_m" (Lm distance, 2021).

$$L_m = (|x_1 - x_2|^m + |y_1 - y_2|^m)^{\frac{1}{m}}$$

This definition of L_m is actually my own. The notation in NIST is "the generalized distance between two points in a plane with point P_1 at (x_1, y_1) and P_2 at (x_2, y_2), is $(|x_1 - x_2|^m + |y_1 - y_2|^m)^{\frac{1}{m}}$" (Lm distance, 2021), which lacks the explanation for

m-norm. On the other hand, although the definition of Wikipedia gives "the Minkowski distance of order p (p-norm distance)" (Distance - Wikipedia, 2021), I believe having a coherent term, L_m, is better. Therefore, I combine the definition in Wikipedia and NIST and optimize them to get a better definition.

Specifically,

When $m = 2$, $L_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. This represents the "Euclidian distance" (Distance - Wikipedia, 2021).

When $m = 1$, $L_1 = |x_1 - x_2| + |y_1 - y_2|$. This represents the "Manhattan distance" (Distance - Wikipedia, 2021).

When $m = \infty$, $L_\infty = \lim_{m \to \infty} (|x_1 - x_2|^m + |y_1 - y_2|^m)^{\frac{1}{m}} = \max(|x_1 - x_2|, |y_1 - y_2|)$. This represents the "Chebyshev distance" (Distance - Wikipedia, 2021).

More generally, let points $A(a_1, a_2, \dots, a_n)$ and $B(b_1, b_2, \dots, b_n)$ be in a n-dimensional space.

$$L_m = \left(\sum_{i=1}^{n} |a_i - b_i|^m \right)^{\frac{1}{m}}$$

Here m is not required to be an integer but should be not less than 1. Otherwise, the triangular inequality doesn't hold. I will prove this below by contradiction.

Assume $m < 1$, and there are three points, $A(0,0)$, $B(0,1)$, and $C(1,1)$. They form a triangle ΔABC. Then the three sides must satisfy the triangular inequality, which is $AB + BC > AC$. In terms of L_m distance, $AB = BC = (0^m + 1^m)^{\frac{1}{m}} = 1$ and $AC =$

$(1^m + 1^m)^{\frac{1}{m}} = (2 \cdot 1^m)^{\frac{1}{m}} = 2^{\frac{1}{m}} > 2^1 = 2$. Since $AB + BC < AC$, the triangular inequality does not hold, contradicting the assumption of $m < 1$. Hence, $m \geq 1$.

Next, I will propose several problems to help explain the applications of generalized distance.

Several Problems to Consider

Problem 1

A mouse on point $A(9,14)$ tries to eat a piece of cheese on point $B(16,8)$. The mouse can go directly towards the cheese. Find the shortest distance that the mice need to travel in order to eat the cheese.

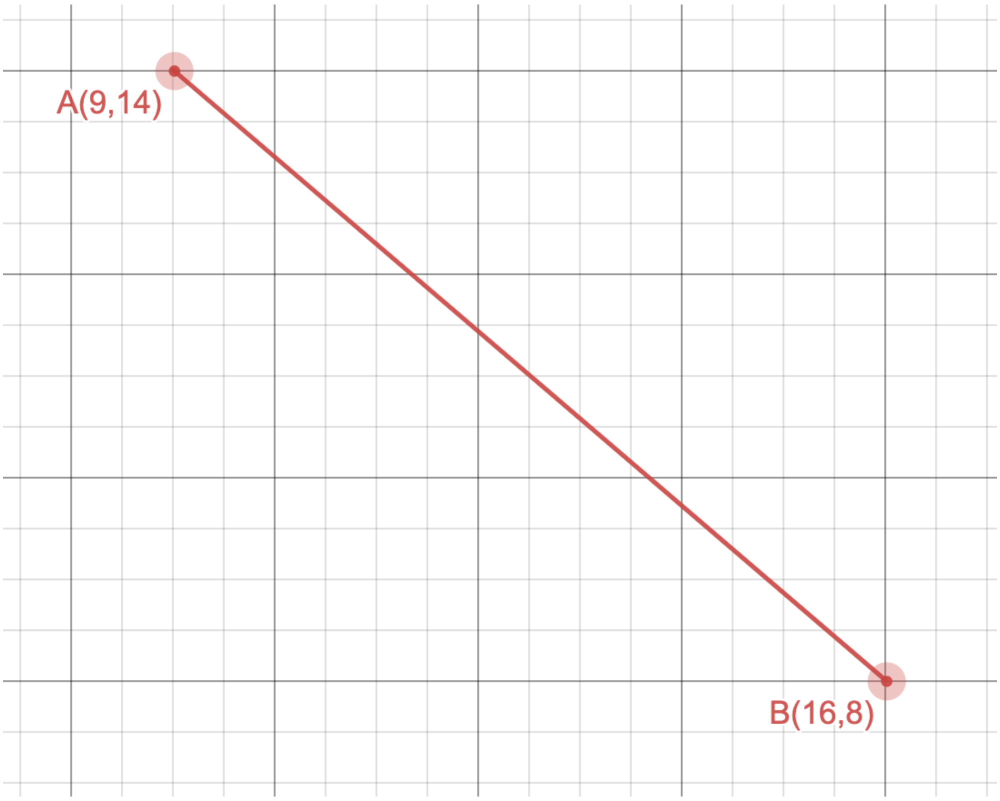

Figure 1. Problem 1 Illustration

Solution

$D = \sqrt{(9-16)^2 + (14-8)^2} = 9.22.$

Note that the procedure of distance calculation is the same with L_2 distance, which is the Euclidian distance. The definition of Euclidian distance is used most widely in our world.

Problem 2

A mouse on point $A(9,14)$ tries to eat a piece of cheese on point $B(16,8)$. The mouse can only travel through the gridlines. Find the shortest distance that the mouse needs to travel in order to eat the cheese.

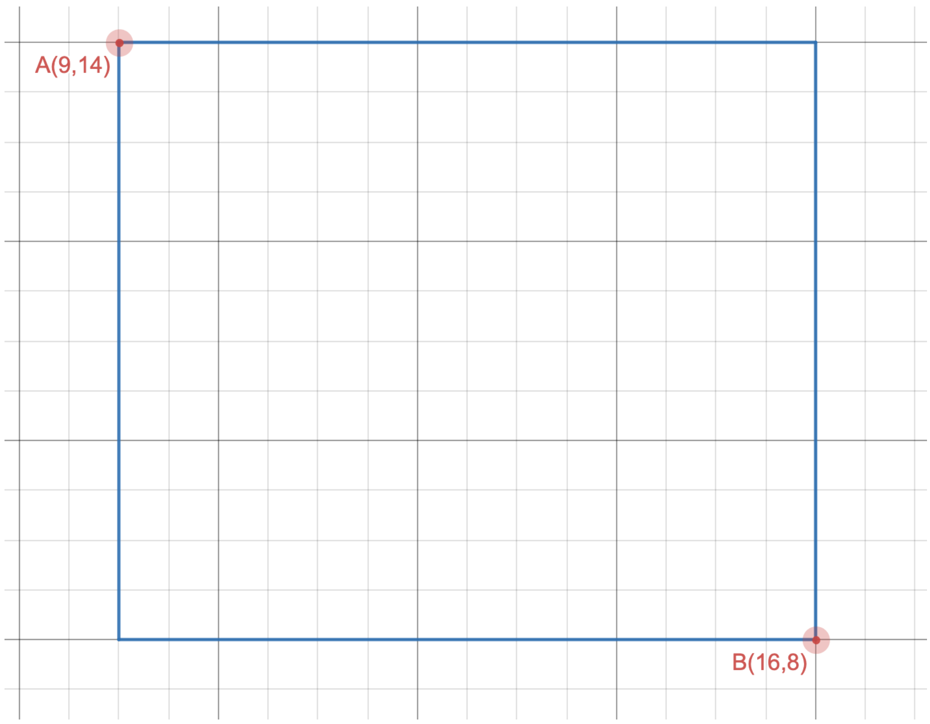

Figure 2. Problem 2 Illustration

Solution

The mice can only travel along the edge and surface of the rectangle. In this case the smallest distance it needs to travel is $D = |16 - 9| + |8 - 14| = 13$.

Note that the format of this distance is similar to L_1. This distance is also called the Manhattan distance or "taxicab-norm" distance because "it is the distance a car would drive in a city laid out in square blocks if there are no one-way streets" (Distance - Wikipedia, 2021). It can also measure "the distance between squares on the chessboard for rooks" (Euclidean vs Manhattan vs Chebyshev Distance, 2021).

Problem 3

Suppose the plane is now a chessboard. A king lies on point $A(9,14)$ and a static enemy lies on $B(16,8)$. The king can travel one unit to any eight directions (up, down, left, right, up-left, up-right, down-left or down-right) to the position next to it. Find the least number of steps that the king needs to move to reach the enemy.

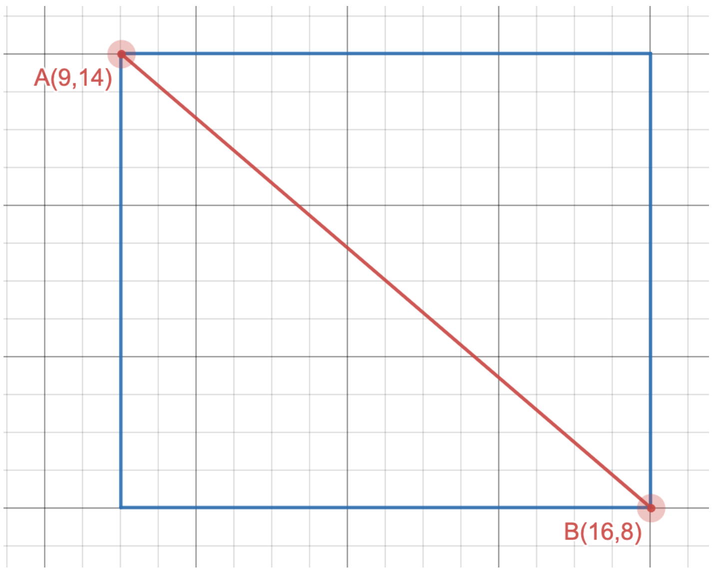

Figure 3. Problem 3 Illustration

Solution

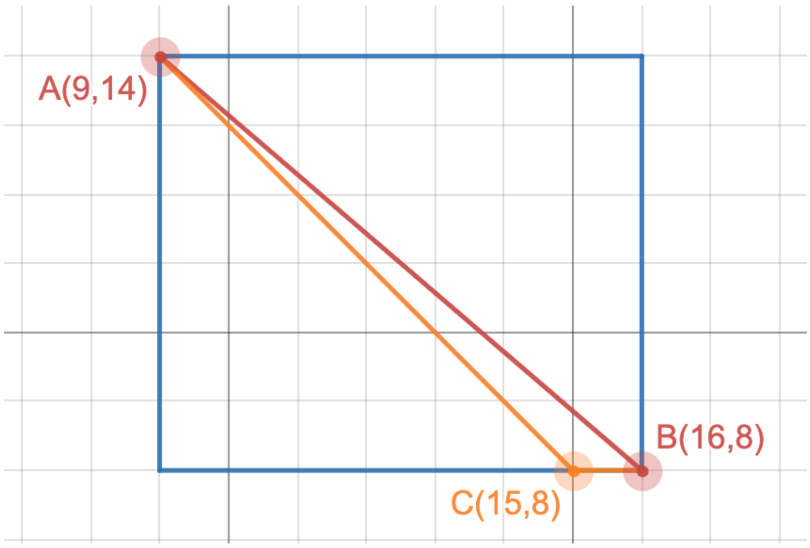

Figure 4. Problem 3 Solution

The king can first go 6 units down-right through the yellow line to $C(15,8)$, then go one unit right to reach $B(16,8)$ and reaches the enemy. The total steps required is $6 + 1 = 7$.

What if the coordinates for the king and the enemy are different, say, $A(x_1, y_1)$ and $B(x_2, y_2)$? Also, I generalize the king movement rule to be: one time it can move n units to displace one of $(n,0), (-n,0), (0,n), (0,-n), (n,n), (n,-n), (-n,n)$ and $(-n,-n)$ units where n is a real positive number.

For the king to reach the enemy, the king's and the enemy's final x coordinates and y coordinates should be equal. Therefore, $|\Delta x|$ steps are required for their x coordinates to equal and $|\Delta y|$ steps are required for their y coordinates to equal.

Since the king can travel diagonally, both $|\Delta x|$ and $|\Delta y|$ can be shortened simultaneously. The king can travel diagonally until the less of $|\Delta x|$ and $|\Delta y|$ is zero, then travel either horizontally or vertically until it reaches the target. Therefore, the less one of $|\Delta x|$ and $|\Delta y|$ have no contribution to the final steps, so the resultant steps required is the larger one of $|\Delta x|$ and $|\Delta y|$. To express it in mathematical language, the minimum steps required is $\max(|\Delta x|, |\Delta y|) = \max(|x_1 - x_2|, |y_1 - y_2|)$.

Note that the format of this distance is similar to L_∞ or the Chebyshev distance. Particularly in two-dimension, it is "the minimum number of moves kings require to travel between two squares on a chessboard" (Distance - Wikipedia, 2021).

Conversion between Chebyshev distance and Manhattan distance

After finding applications of different L_m distances in real life, I notice that L_1 and L_∞ are similar in notation. $L_1 = |x_1 - x_2| + |y_1 - y_2|$ and $L_\infty = \max(|x_1 - x_2|, |y_1 - y_2|)$. I am curious to explore whether there exists some common pattern between L_1 and L_∞.

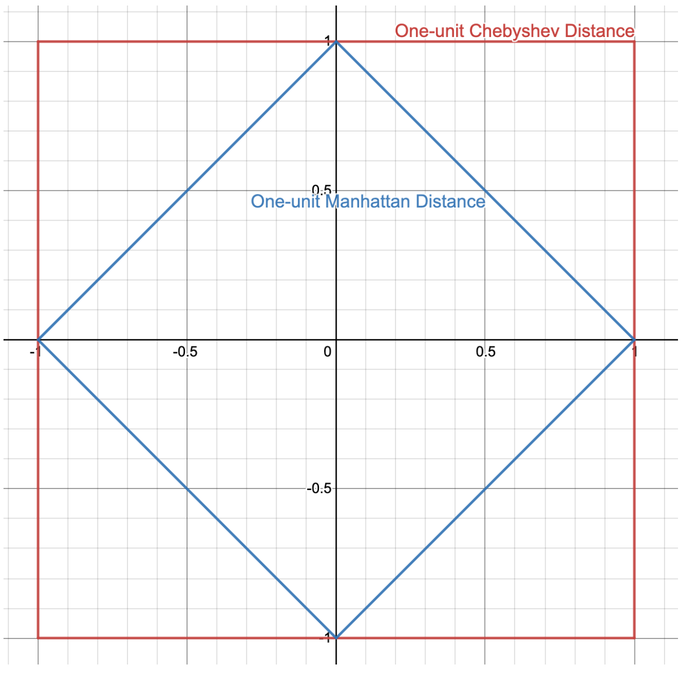

Figure 5. Manhattan and Chebyshev distance relationship in two-dimension

We define the **Manhattan system** as the coordinate system that all the points in this system have their distance to the origin calculated as the **Manhattan distance**, and the **Chebyshev system** as the coordinate system that all the points in this system have their distance to the origin calculated as the **Chebyshev distance**. On Figure 5, the blue square is a "unit square" in the Manhattan system which indicates all the points that have one-unit Manhattan distance to the origin. The red square is a "unit square" in the Chebyshev system which indicates all the points that have one-unit Chebyshev distance to the origin. Therefore, we can use the blue square to represent the Manhattan system and the red square to represent the Chebyshev system, because other points in one system lies on the extended square from the unit square of that system. Consequently, **analyzing the relationship between the two systems is congruent to analyzing the relationship between the two unit squares**.

We define one Manhattan system and one Chebyshev system as "convertible" as: for an arbitrary point (x_M, y_M) in the Manhattan system, there is always a corresponding point (x_C, y_C) in the Chebyshev system so that the Manhattan distance between (x_M, y_M) and the origin equals the Chebyshev distance between (x_C, y_C) and the origin, *vice versa*.

Method 1: Geometric Approach

Since the two unit squares are "similar" (they have the same shape) (Similarity (geometry) - Wikipedia, 2021) with both centers at the origin, we can rotate the blue square $45°$ counterclockwise and enlarge it by scale factor $\sqrt{2}$ to obtain the red square. Reversely, by rotating the red square $45°$ counterclockwise and contract it by scale factor $\sqrt{2}$, we obtain the blue square. So the "rotating and scaling" approach can convert between the Manhattan distance and the Chebyshev distance.

Method 2: Algebraic approach

The concept of shifting coordinate system can also be explained using vector and matrix.

I define the dot product between a two-dimensional vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and a one-dimensional vector $[n]$ as $\begin{bmatrix} an \\ bn \end{bmatrix}$, and the resultant vector has the geometric meaning of having an units on the x-coordinate and bn units on the y-coordinate.

I define a two-by-two transformation matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ be a combined transformation, combining the transformation of $\begin{bmatrix} a \\ b \end{bmatrix}$ on the x-coordiante and $\begin{bmatrix} c \\ d \end{bmatrix}$ on the y-coordinate of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$. Because the x- and y- coordinate of the vector can be considered as one-dimensional vectors, the transformation of $\begin{bmatrix} a \\ b \end{bmatrix}$ on $[x]$ is $\begin{bmatrix} ax \\ bx \end{bmatrix}$ and the transformation $\begin{bmatrix} c \\ d \end{bmatrix}$ on $[y]$ is $\begin{bmatrix} cy \\ dy \end{bmatrix}$. So the resultant vector is $\begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}$.

If we consider a point's coordinate being a vector, and a transformation of coordinate system as a matrix, then by solving the matrix for transformation we can obtain the corresponding point in the other coordinate system.

In two-dimension, let $A(x_1, y_1)$ and $B(x_2, y_2)$, then their Manhattan distance is

$$\begin{aligned} L_1 &= |x_1 - x_2| + |y_1 - y_2| \\ &= \max(x_1 - x_2 + y_1 - y_2, x_1 - x_2 + y_2 - y_1, x_2 - x_1 + y_1 - y_2, x_2 - x_1 + y_2 - y_1) \\ &= \max(|(x_1 + y_1) - (x_2 + y_2)|, |(x_1 - y_1) - (x_2 - y_2)|) \end{aligned}$$

Which is "the Chebyshev distance between $(x_1 + y_1, x_1 - y_1)$ and $(x_2 + y_2, x_2 -$

y_2)" (Juli - OI Wiki, 2021). Therefore, the Manhattan distance of the point (x, y) to the origin in the first system equals the Chebyshev distance of $(x+y, x-y)$ to the origin in the second system. Because $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$, applying a transformation of $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ converts the point (x, y) in the Manhattan system to the point $(x+y, x-y)$ in the Chebyshev system.

Also, the Chebyshev distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$
\begin{aligned}
L_\infty &= \max(|x_1 - x_2|, |y_1 - y_2|) \\
&= \max\left(\left| \frac{x_1 + y_1}{2} - \frac{x_2 + y_2}{2} \right| + \left| \frac{x_1 - y_1}{2} - \frac{x_2 - y_2}{2} \right| \right) \\
&= \left| \frac{x_1 + y_1}{2} - \frac{x_2 + y_2}{2} \right| + \left| \frac{x_1 - y_1}{2} - \frac{x_2 - y_2}{2} \right|
\end{aligned}
$$

Which is "the Manhattan distance between $\left(\frac{x_1+y_1}{2}, \frac{x_1-y_1}{2}\right)$ and $\left(\frac{x_2+y_2}{2}, \frac{x_2-y_2}{2}\right)$" (Juli - OI Wiki, 2021). Therefore, the Chebyshev distance of the point (x, y) to the origin in the first system equals the Manhattan distance of the point $\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ to the origin in the second system. Because $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{bmatrix}$, applying a transformation of $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ converts the point (x, y) in the Chebyshev system to point $\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ in the Manhattan system.

Notice that the first transformation matrix, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, is the second transformation matrix, $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, times two. This is consistent with the scale factor ratio in the geometric approach. In the first case, the blue unit square expands with scale factor $\sqrt{2}$ while in the second case, the red unit square contracts with scale factor $\sqrt{2}$. The combined effect is an expansion with scale factor $\frac{\sqrt{2}}{\frac{1}{\sqrt{2}}} = 2$. This provides an effective proof to my deduction of the transformation matrix.

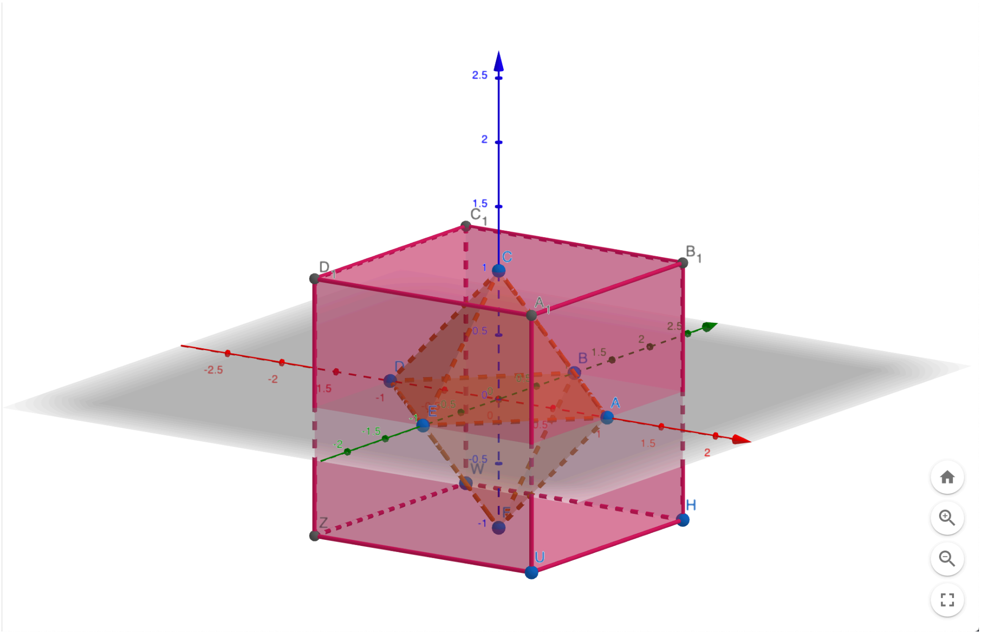

Figure 6. Manhattan and Chebyshev distance relationship in n-dimension

What about higher dimensions? I try on this but do not find much result. Similar to the two-dimension process, in Figure 6 above, the inner orange octahedron represents all the points that have one-unit Manhattan distance from the origin, while the red cube represents all the points that have one-unit Chebyshev distance from the origin. However, octahedron and cube are clearly not similar, so my "rotating and scaling" approach does not work anymore. I really wonder what their relationship is in a higher dimension and I am eager to find better methods for further discovery.

Conclusion

For the generalized L_m distance in Euclidean geometry, it has specific applications in real life when some certain m values are assigned. When $m = 1$, it is the

Manhattan distance that calculates the length that a car needs to travel through a city with grid routines. When $m = 2$, it is the Euclidean distance that calculates the straight-line distance between two points. When $m = \infty$, it is the Chebyshev distance that calculates the minimum steps required for a king in chess to reach its enemy. Surprisingly, the Chebyshev distance and the Manhattan distance in two dimensions can be converted between each other, which I have shown on page 8. The geometric approach provides a more intuitive way of understanding the conversion using unit squares, while the algebraic approach uses transformation matrixes that can deal with data of actual point coordinates. Both methods have advantages when solving different kind of problems.

Certainly, there are limitations with my essay. I did not investigate further L_m distances when m takes values other than 1, 2, and ∞ because other results are related with “super ellipse” (Distance - Wikipedia, 2021), definitely beyond the scope of IB Higher Level math. Additionally, my discovery of the conversion of the Chebyshev distance and the Manhattan distance is limited to two-dimension. I would prefer to investigate further in those extended questions in the future when I have a more profound understanding of those fascinating math topics.

Reference:

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